2020
MATHEMATICS
[HONOURS]
Paper : III
[SUPPLEMENTARY]

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meaning.

1. Answer any **five** questions: 1×5=5

a) Let $\mathbb{R}^3$ and $W = \{(x, y, 0) : x, y \in \mathbb{R}\}$. Then describe the coset (2,2,2) + W.

b) Prove that if G is abelian and $\varphi : G \to G'$ is a homomorphism of G onto G', prove that G' is abelian.

c) If G is a finite group of even order, show that there must be an element $a \neq e$ such that $a = a^{-1}$.

d) Show that $f(x) = x^{\frac{1}{3}}, x \in \mathbb{R}$, is not differentiable at x = 0.

e) Prove that $\lim \frac{1}{2018^{n^2}} = 0$.

f) Give an example of two divergent sequences X and Y such that X + Y is convergent.

g) Given any $x \in \mathbb{R}$, show that there exists a unique $n \in \mathbb{Z}$ such that $n - 1 < x < n$.

h) Show that $|a + b| = |a| + |b|$ iff $ab \geq 0$.

2. Answer any **ten** questions: 2×10=20

a) Prove that if the series $\sum_{n=1}^{\infty} x_n$ converges, then $\lim x_n = 0$.

b) If $x_n = \sqrt{n}$, show that (x_n) satisfies $\lim |x_{n+1} - x_n|$, but that it is not a Cauchy sequence.

c) Determine the limit of the sequence $\left(n^{\frac{1}{n^2}}\right)$.

d) Let $J_n = \left(0, \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} J_n = \phi$.

e) Prove that if y > 0, there exists $n \in \mathbb{N}$ such that $2^{\frac{1}{n}} < y$.

f) Prove that if x and y are real numbers with x < y, then there exists an irrational number z such that x < z < y.

g) Let V be a vector space and W be a subspace of V. Prove that there exists an onto linear map $T : V \to V/W$ such that KerT = W.

h) Find by inspection the eigen values and eigen vectors of $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

i) Give an example of a field of order 9.

j) If p is a prime number a is prime to p then prove that $a^{p^2-p} \equiv 1 \left(\bmod p^2\right)$.

k) Suppose R is an integral domain in which 20.1 = 0 and 12.1=0. Find characteristic of R.

l) Prove that the group ($\mathbb{Z}$, +) has only one non trivial automorphism.

m) If G and G' are two isomorphic groups then prove that Z(G) and $Z(G')$ are isomorphic.

3. Answer any **five** questions: $6\times5 = 30$

a) i) Let (x_n) be a sequence of positive real numbers such that $L = \lim \frac{x_{n+1}}{x_n}$ exists. Prove that if L < 1, then (x_n) converges to 0.

ii) Determine the limit of the sequence $\left(3\sqrt{n}^{\frac{1}{2n}}\right)$. (4+2)

b) i) Let $S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$. Find infimum and supremum of S.

ii) Let S be a set of nonnegative real numbers that is bounded above and let $T = \left\{x^2 : x \in \mathbb{R}\right\}$. Prove that if u = supS, then u^2 = supT. (3 + 3)

c) i) Let (x_n) be a sequence of real numbers that converges to x and suppose that $x_n > 0$. Then the sequence $\sqrt{x_n}$ of positive square roots converges to $\sqrt{x}$.

ii) Give examples of two divergent sequences whose product converges. (4+2)

d) i) Let (x_n) be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of (x_n) converges to x. Then the sequence (x_n) converges to x.

ii) Prove that $\lim_{x\to0} x\cos\left(\frac{1}{x}\right)$ does not exist but that $\lim_{x\to0} x\cos\left(\frac{1}{x}\right) = 0$. (4+2)

e) i) Let A be a n×n matrix over the field of real numbers, such that the sum of the term of A along each row is 1. Show that 1 is a characteristic root of A.

ii) Show that characteristic roots of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

are the fourth roots of unity. (3 + 3)

f) i) Show that it is impossible to find a homomorphism of $\mathbb{Z}_5$ onto $\mathbb{Z}_4$.

ii) If G/Z(G) is cyclic, show that G is abelian. (3+3)

g) i) Let $a_1, a_2, \ldots, a_n$, are unequal positive real numbers in AP, show that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots, + \frac{1}{a_n} > \frac{2a_n}{a_1 + a_n}$$

ii) Define minimal polynomial and prove that it is unique. (3+3)

h) i) State and prove Bolzano's Intermediate Value Theorem.

ii) Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $f\{x) > 0$ for each $x \in [a,b]$. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in [a,b]$. (4+2)

i) i) If $f : A \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is a Cauchy

sequence in A, then prove that $f(x_n)$ is a Cauchy sequence in $\mathbb{R}$.

ii) Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on the set $[a, \infty]$, where a is a positive constant. (4+2)

4. Answer any **three**: 15×3 =45

a) i) Let R be a ring in which $x^3 = x$ for every $x \in R$. Prove that R is commutative.

ii) Prove that a group of order 9 must be abelian.

iii) Let $\{v_i\}_{i=1}^{k}$ be a set of pairwise orthogonal vectors in an inner product space V. Then prove that

$$\left\| \sum_{i=1}^{k} v_i \right\|^2 = \sum_{i=1}^{k} \|v_i\|^2$$

iv) Find four consequtive integers divisible by 3,4,5,7 respectively. (4+3+4+4)

b) i) Let $T : V \to V$ be a symmetric linear map on a finite dimensional inner product saace V. Prove that there exists an orthonormal basis of V consisting of eigen vectors.

ii) Apply (i) to prove that if A is a real symmetric to n×n matrix. Then there exists an orthogonal matrix B such that BAB^{-1} is a diagonal matrix whose entries are the eigen values of A.

iii) Reduce the following quadratic $\mathbb{R}^3$ into standard form and determine:

$$f(x,y,z) \equiv 4xz + 4y^2 + 8y + 8 = 0$$

(7+3+5)

c) i) List all elements of $\mathbb{Z}_{40}$ which have order 10.

ii) Let G be an abelian group and p be a prime. Show that the set of all elements whose orders are powers of p is a subgroup of G.

iii) Prove that

$H = \{A \in Gl(2,\mathbb{R}) : |A| \text{is rational}\}$ is rational}. Prove or disprove that H is a subgroup of G. What if we replace rational by integer?

iv) Find two elements in a rings such that both a and b are zero divisors, $a + b \neq 0$, and a + b is not a zero divisor.

(4+4+4+3)

d) i) Let $s_1 > 0$; and $s_{n+1} = \frac{1}{2}\left(s_n + \frac{16}{s_n}\right)$. Prove that (s_n) converges to 4.

ii) Establish the convergence or the divergence of the sequence (y_n); where

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} .... + \frac{1}{2n}.$$

iii) Find the limit of the following sequence $\left(\left(1+\frac{1}{n}\right)^{2n}\right)$. (5+5+5)

e) i) A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on [a, b].

ii) Show that the function $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

iii) Let $f : [0,1] \to \mathbb{R}$ be a continuous function that does not take on any of its values twice and $f(0) < f(1)$. Show that f is strictly increasing on [0,1]. (5+5+5)

f) i) A monotone sequence of real numbers is convergent if and only if it is bounded.

ii) Use monotone convergency theorem to show that the series $\sum_{n\in\mathbb{N}} \frac{1}{n}$ is divergent.

iii) Let (z_n) be a sequence of real numbers defined by $z_1 = 1$ and $z_{n+1} = \sqrt{2z_n}$. Determine whether the sequence is converging. If the sequence converges find the limit. (5+5+5)

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